Three remarks on one dimensional bi-Lipschitz conjugacies

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Abstract. In this Note we deal with bi-Lipschitz homeomorphisms conjugating actions by C^r circle diffeomorphisms. Using an equivariant version of the classical Gottschalk-Hedlund Lemma, we prove that such a homeomorphism is necessarily a C^r diffeomorphism if these actions are non free, minimal, and ergodic with respect to the Lebesgue measure. However, we exhibit a large variety of examples showing that this is far from being true if the actions are non minimal. This clarifies slightly the content of a classical result by Ghys and Tsuboi, who proved that, roughly, C^1 conjugacies between non free C^r one-dimensional dynamical systems are automatically of class C^r. All the results of this Note are contained in [6].

Introduction

Let θ_1 and θ_2 be two non free actions of a finitely generated group Γ by C^r circle diffeomorphisms, where $r \geq 1$. Suppose that there exists some bi-Lipschitz homeomorphism $\phi : \mathrm{S}^1 \to \mathrm{S}^1$ conjugating θ_1 and θ_2, *i.e.* such that the equality $\phi \circ \theta_1(g) = \theta_2(g) \circ \phi$ holds for every $g \in \Gamma$. The problem we deal with in this Note is the following: under which conditions on θ_1 (and θ_2) the map ϕ is automatically a C^r diffeomorphism? This is much inspired by the classical work [5] by Ghys and Tsuboi, where the same question is addressed for C^1 conjugacies ϕ assuming that $r \geq 2$. In that context they proved that ϕ is necessarily a C^r diffeomorphism if there is no finite orbit; if there are finite orbits, then ϕ is a C^r diffeomorphism restricted to the complementary set of these orbits. See also proposition 4.9 in [1] for a closely related result in the $C^{1+\alpha}$ case.

For the non free case we show in this Note that the situation is quite different when ϕ is only assumed to be bi-Lipschitz: in general, if the actions are minimal then ϕ is still smooth, but for the non minimal case there are a lot of bi-Lipschitz non smooth conjugacies.

Theorem A. *Let θ_1 and θ_2 be two minimal non free actions of a finitely generated group by C^r circle diffeomorphisms, where $r \geq 1$. If θ_1 and θ_2 are conjugated by a bi-Lipschitz circle homeomorphism ϕ and are ergodic with respect to the Lebesgue measure, then ϕ is a C^r diffeomorphism.*

The proof of this theorem uses a version of the classical Gottschalk-Hedlund Lemma for group actions. Although such a version does not appear in the literature, its proof is an easy modification of the classical one. We decided to include it here for the convenience of the reader and because of its simplicity and beauty.

Concerning the hypothesis of ergodicity, it is conjectured that minimal actions of finitely generated groups by C^r circle diffeomorphisms are always ergodic with respect to the Lebesgue measure for $r \geq 2$. For $r \in]1,2[$ the situation is more complicated: if the action is non free then the same should be true, but there seem to be a lot of free minimal non ergodic actions (compare with [8]). Finally, for $r = 1$ there are minimal non ergodic actions both in the free [8] and the non free cases (these last ones can be constructed using the examples given in [9]).

Let us now consider the non minimal case. Note that a conjugacy of an action to it-self is a map which centralizes this action. Moreover, if θ_1 and θ_2 are two actions by C^r circle diffeomorphisms which are supposed *a priori* to be conjugate by some C^r diffeomorphism ϕ_0, and if ϕ is any

other bi-Lipschitz homeomorphism conjugating them, then the bi-Lipschitz homeomorphism $\phi_0^{-1}\phi$ centralizes θ_1. This is why it is so important to study the centralizer problem before dealing with the general conjugacy problem. At this level we prove the following result.

Theorem B. *Let Γ be any finitely generated group of C^2 circle diffeomorphisms whose action is non minimal and for which the stabilizers of points are either trivial or infinite cyclic. Then there exists a bi-Lipschitz circle homeomorphism which is not C^1 and which commutes with every element of Γ. Moreover, such a homeomorphism can be taken to be non differentiable on every open interval of the circle.*

The hypothesis on stabilizers is not very strong. For instance, it is always satisfied for real-analytic non minimal actions without finite orbits. (This result is due to Hector; a complete proof appears in the Appendix of [7].) Of course, it is also satisfied by many other smooth non real-analytic interesting actions. Without this hypothesis it is easy to see that, in some cases, bi-Lipschitz conjugacies are forced to be smooth.

We finish with an example where the conjugacy problem cannot be reduced (in a very strong sense) to a problem of centralizers. It would be interesting to know if the examples of the following theorem can be real-analytic.

Theorem C. *There exist two finitely generated groups of C^∞ circle diffeomorphisms acting non freely and without finite orbits which are bi-Lipschitz conjugate but for which there is no C^1 circle diffeomorphism conjugating them.*

In what follows we will consider only orientation preserving maps, but the results can be easily extended to the non orientation preserving case (we leave this as a task to the reader). Moreover, by using standard methods, the results of this Note can be generalized into the context of codimension one foliations or general one-dimensional pseudo-groups.

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1 The minimal case

1.1 A Gottschalk-Hedlund Lemma for group actions

Let X be a compact metric space and Γ a finitely generated group acting on it by homeomorphisms. A *cocycle* associated to this action is a map $c : \Gamma \times X \to \mathbb{R}$ such that for each fixed $f \in \Gamma$ the map $x \mapsto c(f,x)$ is continuous, and such that for every f, g in Γ and every $x \in X$ one has

$$c(fg,x) = c(g,x) + c(f,g(x)). \tag{1}$$

Lemma 1.1. *Suppose that the Γ-action on X is minimal. Then the following are equivalent:*
(i) *there exists some $x_0 \in X$ and some constant $C > 0$ such that $|c(f,x_0)| \leq C$ for every $f \in \Gamma$,*
(ii) *there exists some continuous function $\varphi : X \to \mathbb{R}$ such that $c(f,x) = \varphi(f(x)) - \varphi(x)$ for all $f \in \Gamma$ and all $x \in X$.*

Proof. If the second condition is satisfied then

$$|c(f,x_0)| \leq |\varphi(f(x_0))| + |\varphi(x_0)| \leq 2\|\varphi\|_{C^0},$$

which proves the validity of condition (i).

Reciprocally, let us suppose that the first condition holds. For each $f \in \Gamma$ consider the homeomorphism $\hat{f}$ of the space $X \times \mathbb{R}$ defined by $\hat{f}(x,t) = (f(x), t + c(f,x))$. It is easy to see that the cocycle relation (1) implies that this defines a group action of Γ on $X \times \mathbb{R}$, in the sense that $\hat{f}\hat{g} = \widehat{fg}$ for all f, g in Γ. Moreover, condition (i) implies that the orbit of the point $(x_0, 0)$ under this action is bounded; in particular, its closure is a (non empty) compact invariant set. Using Zorn's lemma, one easily deduces the existence of a minimal non empty compact invariant subset M of $X \times \mathbb{R}$. We claim that this subset is the graph of a continuous function from X to $\mathbb{R}$.

First of all, since the action of Γ on X is minimal, the projection of M on X is the whole space. Moreover, if $(\bar{x}, t_1)$ and $(\bar{x}, t_2)$ belong to M for some $\bar{x} \in X$ and some $t_1 \neq t_2$, then this implies that $M \cap M_t \neq \emptyset$, where $t = t_2 - t_1 \neq 0$ and $M_t = \{(x, s+t) : (x,s) \in M\}$. Note that the Γ-action on $X \times \mathbb{R}$ commutes with the map $(x,s) \mapsto (x, s+t)$; in particular, M_t is also invariant. But since M is minimal, this implies that $M = M_t$. One then concludes that $M = M_t = M_{2t} = \ldots$, which is impossible since M is compact.

We have then proved that for every $x \in X$ the set M contains exactly one point of the form (x,t). Putting $\varphi(x) = t$ one obtains a function form X to $\mathbb{R}$, which is continuous, since its graph (which coincides with M) is compact.

Finally, since the graph of φ is invariant by the action, for all $f \in \Gamma$ and all $x \in X$ the point $\hat{f}(x, \varphi(x)) = (f(x), \varphi(x) + c(f,x))$ must be of the form $(f(x), \varphi(f(x)))$, which implies that $c(f,x) = \varphi(f(x)) - \varphi(x)$.

Lemma 1.2. *Let X be a compact metric space and Γ a finitely generated group acting on it by homeomorphisms. Suppose that the Γ-action on X is minimal and ergodic with respect to some probability measure μ, and let c be a cocycle associated to this action. If φ is a function in $L^\infty_\mu(X)$ such that for all $f \in \Gamma$ and μ almost every $x \in X$ one has*

$$c(f,x) = \varphi(f(x)) - \varphi(x), \tag{2}$$

then there exists some continuous function $\tilde{\varphi} : X \to \mathbb{R}$ which coincides μ a.e. with φ and such that for all $f \in \Gamma$ and all $x \in X$ one has

$$c(f,x) = \tilde{\varphi}(f(x)) - \tilde{\varphi}(x). \tag{3}$$

Proof. Let Y_0 be the set of points in which (2) does not hold for some $f \in \Gamma$. Since Γ is finitely generated, $\mu(Y_0) = 0$. Let Y_1' the complementary set of the essential support of φ, and let $Y_1 = \cup_{f \in \Gamma} f(Y_1')$. Take a point x_0 in the full measure set $X \setminus (Y_0 \cup Y_1)$. Equation (2) then gives $|c(f,x_0)| \leq 2\|\varphi\|_{L^\infty}$ for all $f \in \Gamma$. By the preceding lemma, there exists some continuous function $\tilde{\varphi} : X \to \mathbb{R}$ such that (3) holds for every x and f. This implies that μ a.e. we have

$$\tilde{\varphi} \circ f - \tilde{\varphi} = \varphi \circ f - \varphi,$$

and so

$$\tilde{\varphi} - \varphi = (\tilde{\varphi} - \varphi) \circ f.$$

Since the Γ-action on X is assumed to be μ-ergodic, the difference $\tilde{\varphi} - \varphi$ has to be μ a.e. constant. Finally, changing $\tilde{\varphi}$ by some $\tilde{\varphi} + C$, we may force this constant to be equal to zero.

1.2 Proof of Theorem A

Note that if ϕ is a bi-Lipschitz homeomorphism of the circle conjugating the actions θ_1 and θ_2 of our group Γ, then ϕ and ϕ^{-1} are almost everywhere differentiable with L^∞ functions as derivatives.

Therefore, the function $x \mapsto \log(\phi'(x))$ is also L^∞. The relation $\theta_1(f) = \phi^{-1} \circ \theta_2(f) \circ \phi$ gives almost everywhere

$$\log(\theta_1(f)'(x)) = \log(\phi'(x)) - \log(\phi'(\theta_1(f)(x))) + \log(\theta_2(f)'(\phi(x))).$$

Putting $\varphi = -\log(\phi')$ and $c(f,x) = \log(\theta_1(f)'(x)) - \log(\theta_2(f)'(\phi(x)))$ this gives, for all $f \in \Gamma$ and almost every $x \in \mathrm{S}^1$,

$$c(f,x) = \varphi(\theta_1(f)(x)) - \varphi(x).$$

One easily checks the cocycle relation

$$c(fg,x) = c(g,x) + c(f, \theta_1(g)(x)).$$

Since the θ_1-action is supposed to be ergodic, Lemma 1.2 gives the existence of a continuous function $\tilde{\varphi}$ which coincides almost everywhere with φ and such that (3) holds for every x and f. By integrating, one concludes that the derivative of ϕ is well defined everywhere and coincides with $\exp(-\tilde{\varphi})$. In particular, ϕ is of class C^1, and interviewing the roles of θ_1 and θ_2, one concludes that ϕ is a C^1 diffeomorphism. In order to prove that ϕ is a C^r diffeomorphism, one can use the main result of [5] for $r \geq 2$, as well as Proposition 4.4 of [1] for the $C^{1+\alpha}$ case.

2 The non minimal case

2.1 Non smooth bi-Lipschitz centralizers

Before passing to the proof of Theorem B, let us explain the main idea by giving a very simple and general construction (which seems to be well known to the specialists) of a non smooth bi-Lipschitz homeomorphism centralizing an interval diffeomorphism without interior fixed points.

Let f be a C^2 diffeomorphism of $I = [a,b]$ such that $f^n(x)$ converges to a as n goes to infinity for every $x \in [a,b[$. Fix any point $c \in]a,b[$, and consider any bi-Lipschitz homeomorphism h from the interval $[f(c),c]$ to itself. Extending h to $]a,b[$ in such a way that $fh = hf$, and then putting $h(a) = a$ and $h(b) = b$, we obtain a well defined self-homeomorphism of $[a,b]$ (still denoted by h). We claim that this globally defined h is still bi-Lipschitz. More precisely, if M is a bi-Lipschitz constant for h on $[f(c),c]$, then Me^V is a bi-Lipschitz constant for h on $[a,b]$, where V is the total variation of the logarithm of the derivative of f:

$$V = var(\log(f')) = \sup_{a \leq a_0 \leq a_1 \leq \ldots \leq a_n \leq b} \sum_{i=0}^{n-1} |\log(f'(a_{i+1})) - \log(f'(a_i))| = \int_a^b \left| \frac{f''(s)}{f'(s)} \right| ds.$$

Indeed, let us suppose for instance that x belongs to $f^n([f(c),c])$ for some $n \geq 0$, and that h has a well defined derivative at the point $f^{-n}(x) \in [f(c),c]$ which is less or equal than M. (Note that this is the case for almost every $x \in [f^{n+1}(c), f^n(c)]$.) Because of the relation $h = f^n h f^{-n}$ one has the inequality

$$h'(x) = h'(f^{-n}(x)) \cdot \frac{(f^n)'(hf^{-n}(x))}{(f^n)'(f^{-n})(x)} \leq M \cdot \frac{(f^n)'(hf^{-n}(x))}{(f^n)'(f^{-n})(x)}. \tag{4}$$

Now, putting $y = f^{-n}(x) \in [f(c),c]$ and $z = h(y) \in [f(c),c]$, we have

$$\left| \log\left(\frac{(f^n)'(z)}{(f^n)'(y)} \right) \right| = \left| \log\left(\frac{\prod_{i=0}^{n-1} f'(f^i(z))}{\prod_{i=0}^{n-1} f'(f^i(y))} \right) \right| \leq \sum_{i=0}^{n-1} \left| \log(f'(f^i(z))) - \log(f'(f^i(y))) \right| \leq V.$$

Introducing this last inequality into (4) one obtains $h'(x) \leq Me^V$. Since x was a generic point, this shows that h has Lipschitz constant bounded by Me^V. The very same argument can be used to check a simlar bound for the Lipschitz constant of h^{-1}.

For the proof of Theorem B we will try to perform an analoous construction. For simplicity, we will give a complete proof only for the first claim of the theorem, leaving to the reader the task of adapting our arguments to prove the second (and stronger) claim concerning the non differentiability on every open interval for some centralizing bi-Lipschitz homeomorphism.

Let us start by recalling that if Γ is group of C^2 circle diffeomorphisms (and more generally of circle homeomorphisms) whose action is non minimal, then there are two possibilities: either Γ preserves a minimal Cantor set (called the *exceptional* minimal set), or Γ has finite orbits [2]. Let us consider the first case, which is dynamically more interesting. Fix any connected component $]a, b[$ of the complementary of the exceptional minimal set. By a result due to Hector, the stabilizer in Γ of $I = [a, b]$ is non trivial (see Lemma 2.7 in [3]), and so by the hypothesis of the theorem it is infinite cyclic. Fix a generator f for this stabilizer. If the restriction of f to I is trivial we let h be any bi-Lipschitz non C^1 homeomorphism of I. If not, fix $[\bar{a}, \bar{b}] \subset [a, b]$ such that $f^n(x) \neq x$ for every $x \in]\bar{a}, \bar{b}[$, and $f(\bar{a}) = \bar{a}$ and $f(\bar{b}) = \bar{b}$. Changing f by f^{-1} if necessary, we may assume that $f^n(x)$ converges to $\bar{a}$ as n goes to infinity for every $x \in [\bar{a}, \bar{b}[$. As before consider any point $\bar{c}$ in $]\bar{a}, \bar{b}[$, and consider any bi-Lipschitz non C^1 homeomorphism h of $[f(\bar{c}), \bar{c}]$. This homeomorphism extends in a unique way to a bi-Lipschitz homeomorphism of $[a, b]$ commuting with the restriction of f to $[\bar{a}, \bar{b}]$ and which is the identity on $I \setminus [\bar{a}, \bar{b}]$.

By the hypothesis on stabilizers, it is easy to see that there exists a unique extension of h into a circle homeomorphism (still denoted by h) which commutes with (every element of) Γ and coincides with the identity in the complementary set of $\cup_{g \in \Gamma} g(]a, b[)$. We claim that this extension is still bi-Lipschitz. More precisely, fixing a finite system $\mathcal{G} = \{g_1, \ldots, g_k\}$ of generators of Γ, denoting by V the supremum for the variation of the logarithm of the derivatives of these generators, and choosing a bi-Lipschitz constant M for h on $[a, b]$, we claim that h has bi-Lipschitz constant smaller or equal than Me^{kV} over the whole circle. The proof of this claim is similar to that of the case of the interval (*i.e.* the one given at the beginning of this Section). Let us choose for instance a point $x \in \cup_{g \in \Gamma} (g(I) \setminus I)$, and let's try to estimate $h'(x)$. To do this, let's take a minimal $n \in \mathbb{N}$ for which there exists some $g = g_{i_n} \circ \ldots g_{i_1} \in \Gamma$ with each g_{i_j} belonging to $\mathcal{G}$ and such that $g(x) \in I$. The minimality of n implies that the intervals $I, g_{i_n}^{-1}(I), g_{i_{n-1}}^{-1} g_{i_n}^{-1}(I), \ldots, g_{i_1}^{-1} \cdots g_{i_n}^{-1}(I)$ have disjoint interiors. Using the relation $h = g^{-1}hg$ one obtains, for a generic $x \in g^{-1}(I)$,

$$h'(x) = h'(g(x)) \cdot \frac{g'(x)}{g'(h(x))} \leq M \cdot \frac{g'(x)}{g'(y)}, \tag{5}$$

where $y = h(x) \in g^{-1}(I)$. Then using only the fact that the total variation for the logarithm of the derivative of each g_i is bounded by V, one obtains

$$\left| \log\left(\frac{g'(x)}{g'(y)}\right) \right| \leq \sum_{j=0}^{n-1} |\log(g'_{i_{j+1}}(g_{i_j} \cdots g_{i_1})(x)) - \log(g'_{i_{j+1}}(g_{i_j} \cdots g_{i_1})(y))| \leq \sum_{i=1}^{k} var(\log(g'_i)) \leq kV.$$

Therefore, from (5) one concludes that $h'(x) \leq Me^{kV}$, as desired.

Let us now consider the case of finite orbits. If Γ is finite then consider any bi-Lipschitz non differentiable circle homeomorphism commuting with its (finite order) generator. If Γ is infinite, then because of Hölder and Denjoy Theorems the action of Γ cannot be free. Take a non trivial element $f \in \Gamma$ having fixed points, and let I be some connected component of the complementary set of the union of the finite orbits. Note that f must fix all the points of these orbits. So, proceeding as in the previous case with I and f one can construct a bi-Lipschitz non differentiable circle homeomorphism centralizing Γ.

2.2 Bi-Lipschitz conjugate actions which are non C^1 conjugate

Before entering into the proof of Theorem C, we would like to insist on the fact that the constructions we propose are rather artificial, and definitively it would be much more interesting to give real-analytic examples of groups sharing a similar conjugacy property.

Let us begin by considering a very simple action on the interval illustrating the main idea. For this, let us fix a sequence $(\ell_n)_{n\in\mathbb{Z}}$ of positive real numbers such that ℓ_n/ℓ_{n+1} converges to 1 as $|n|$ goes to infinity, such that $\ell_{2n} = \ell_{2n+1}$ for every $n \in \mathbb{Z}$, and such that $\sum_{n\in\mathbb{Z}} \ell_n = 1$. Then define another sequence $(\bar{\ell}_n)_{n\in\mathbb{Z}}$ by $\bar{\ell}_{2n} = 4\ell_n/3$ and $\bar{\ell}_{2n+1} = 2\ell_{2n+1}/3$. Note that $\sum_{n\in\mathbb{Z}} \bar{\ell}_n = 1$.

For each $n \in \mathbb{Z}$ consider a diffeomorphism f_n from the interval

$$I_n = \Big[\sum_{i<n} \ell_i, \sum_{i\le n} \ell_i\Big]$$

to it-self without interior fixed points. Let ϕ_0 be the homeomorphism of $[0,1]$ whose restriction to each I_n is the affine map sending I_n to

$$\bar{I}_n = \Big[\sum_{i<n} \bar{\ell}_i, \sum_{i\le n} \bar{\ell}_i\Big],$$

and let $\bar{f}_n$ be the diffeomorphism of $\bar{I}_n$ defined by $\bar{f}_n = \phi_0 f_n \phi_0^{-1}$. It is easy to see that if the maps f_n are well chosen (for instance, if they are infinitely tangent to the identity at the extreme points and their C^r norm converge to zero exponentially fast as $|n|$ goes to infinity for every $r \ge 2$), then the map f defined by $f(x) = f_n(x)$ for every $x \in I_n$ and $f(0) = 0$ and $f(1) = 1$, as well as $\bar{f} = \phi_0 f \phi_0^{-1}$, are C^∞ diffeomorphisms of $[0,1]$ which are infinitely tangent to the identity at the extreme points. Moreover, it follows from the definitions that ϕ_0 is a bi-Lipschitz homeomorphism conjugating them. We claim however that there is no C^1 diffeomorphism conjugating f and $\bar{f}$. Indeed, for every homeomorphism ϕ conjugating f and $\bar{f}$ there exists a fixed $N \in \mathbb{N}$ such that for $\phi(I_n) = \bar{I}_{n+N}$ for all $n \in \mathbb{Z}$. If such ϕ was of class C^1 then using the continuity of ϕ' at 1 one could conclude that, as $n \to \infty$,

$$\frac{|\bar{I}_{n+N}|}{|I_n|} \longrightarrow \phi'(1).$$

However, the left hand side expression does not converge. Indeed, if N is even then as $n \to \infty$ one has

$$\frac{|\bar{I}_{2n+N}|}{|I_{2n}|} \longrightarrow \frac{4}{3} \qquad \text{and} \qquad \frac{|\bar{I}_{2n+1+N}|}{|I_{2n+1}|} \longrightarrow \frac{2}{3},$$

whereas if N is odd then as $n \to \infty$ one has

$$\frac{|\bar{I}_{2n+N}|}{|I_{2n}|} \longrightarrow \frac{2}{3} \qquad \text{and} \qquad \frac{|\bar{I}_{2n+1+N}|}{|I_{2n+1}|} \longrightarrow \frac{4}{3}.$$

Now in order to obtain an example with an exceptional minimal set we will try to "glue" the preceding construction in one of the connected components of the complement of such a minimal set. To be more precise, let us consider the injection $\theta : G \to \mathrm{Diff}_+^\infty(\mathrm{S}^1)$ of the Thompson group G obtained by the method of §III.1 of [4] by using a map satisfying the properties (I), (II) and $(\mathrm{III})_\infty$ therein, and having an interval of fixed points. The corresponding action admits an exceptional minimal set, and we can fix an interval I contained in one of the connected components J of the complement of this set in such a way that the restriction to I of every element of G fixing J coincides with the identity map. Let ϕ_I be the affine map sending $[0,1]$ to I, and let $h \in \mathrm{Diff}_+^\infty(\mathrm{S}^1)$ (resp. $\bar{h}$) be defined by $h(x) = \phi_I f \phi_I^{-1}(x)$ for $x \in I$ and $h(x) = x$ for $x \notin I$ (resp. $\bar{h}(x) = \phi_I \bar{f} \phi_I^{-1}(x)$ for $x \in I$ and $\bar{h}(x) = x$ for $x \notin I$). Now consider the induced group Γ which is a quotient of the free product between G and $\mathbb{Z}$. This group has two actions θ_1 and θ_2 by C^∞ circle diffeomorphisms, depending if we choose h or $\bar{h}$ as the generator of $\mathbb{Z}$. These actions are clearly bi-Lipschitz conjugate, but as before it is easy to see that they are non C^1 conjugate.

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